Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## November 2021

## Mathematics: analysis and approaches

## Higher level

## Paper 1

38 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

**1 General**

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

**2 Method and Answer/Accuracy marks**

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

| | Correct answer seen | Further working seen | Any FT issues? | Action |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | 5.65685... *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | 0.468111… (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

## 3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

## 4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This

includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

**7 Alternative forms**

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1,9$ or $1000$ and $1,000$ and $1.000$.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

**8 Format and accuracy of answers**

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

## Section A

**1. METHOD 1**

recognition that $y = \int \cos\left(x - \frac{\pi}{4}\right) \mathrm{d}x$ ***(M1)***

$y = \sin\left(x - \frac{\pi}{4}\right)(+c)$ ***(A1)***

substitute both $x$ and $y$ values into their integrated expression including $c$ ***(M1)***

$2 = \sin\frac{\pi}{2} + c$

$c = 1$

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

***[4 marks]***

**METHOD 2**

$\int_{2}^{y} \mathrm{d}y = \int_{\frac{3\pi}{4}}^{x} \cos\left(x - \frac{\pi}{4}\right) \mathrm{d}x$ ***(M1)(A1)***

$y - 2 = \sin\left(x - \frac{\pi}{4}\right) - \sin\frac{\pi}{2}$ ***A1***

$y = \sin\left(x - \frac{\pi}{4}\right) + 1$ ***A1***

***[4 marks]***

**2.** (a) (i) $x = 3$ ***A1***

(ii) $y = -2$ ***A1***

***[2 marks]***

(b) (i) $(-2, 0)$ (accept $x = -2$) ***A1***

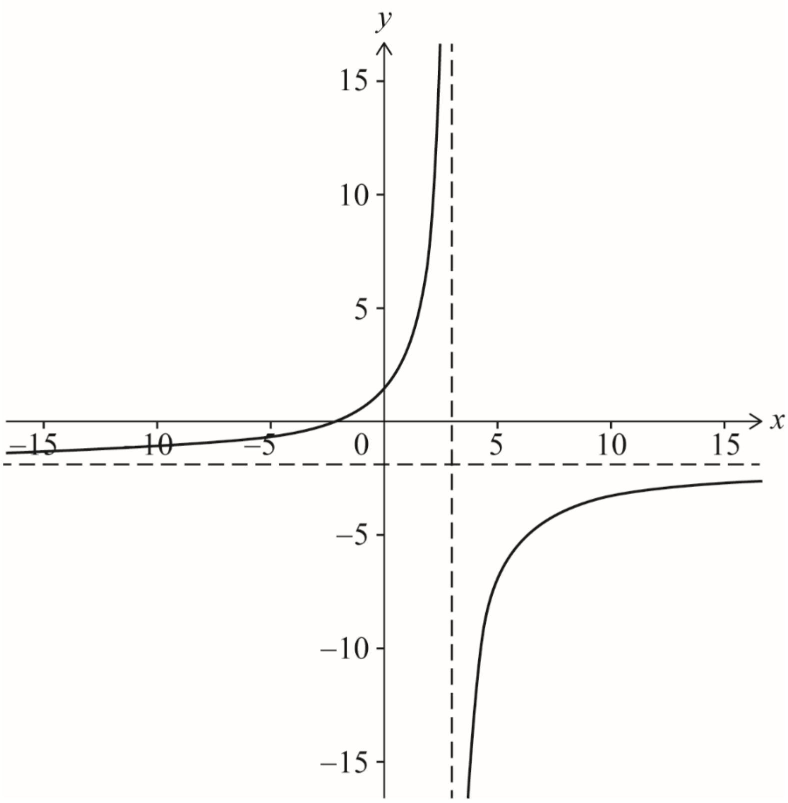
(ii) $\left(0, \frac{4}{3}\right)$ (accept $y = \frac{4}{3}$ and $f(0) = \frac{4}{3}$) ***A1***

***[2 marks]***

*continued…*

*Question 2 continued.*

(c)




***A1***

| |
|---|
| **Note:** Award ***A1*** for completely correct shape: two branches in correct quadrants with asymptotic behaviour. |

***[1 mark]***

*continue…*

*Question 2 continued.*

(d) **METHOD 1**

$$\left(g(x) =\right) y = \frac{ax+4}{3-x}$$

attempt to find $x$ in terms of $y$ ***(M1)***

***OR*** exchange $x$ and $y$ and attempt to find $y$ in terms of $x$

$3y - xy = ax + 4$ ***A1***

$ax + xy = 3y - 4$

$x(a+y) = 3y - 4$

$$x = \frac{3y-4}{y+a}$$

$g^{-1}(x) = \dfrac{3x-4}{x+a}$ ***A1***

| **Note:** Condone use of $y =$ |
|---|

$g(x) \equiv g^{-1}(x)$

$$\frac{ax+4}{3-x} \equiv \frac{3x-4}{x+a}$$

$\Rightarrow a = -3$ ***A1***

***[4 marks]***

*continue…*

*Question 2 continued.*

**METHOD 2**

$g\left(x\right)=\frac{ax+4}{3-x}$

attempt to find an expression for $g\left(g\left(x\right)\right)$ and equate to $x$ ***(M1)***

$$gg\left(x\right)=\frac{a\left(\frac{ax+4}{3-x}\right)+4}{3-\left(\frac{ax+4}{3-x}\right)}=x$$ ***A1***

$$\frac{a\left(ax+4\right)+4\left(3-x\right)}{\left(9-3x\right)-\left(ax+4\right)}=x$$

$$\frac{a\left(ax+4\right)+4\left(3-x\right)}{5-\left(3+a\right)x}=x$$

$$a\left(ax+4\right)+4\left(3-x\right)=x\left(5-\left(3+a\right)x\right)$$ ***A1***

equating coefficients of $x^2$ (or similar)

$a=-3$ ***A1***

***[4 marks]***

***Total [9 marks]***

**3.** attempt to use change the base ***(M1)***

$$\log_3 \sqrt{x} = \frac{\log_3 2}{2} + \log_3\left(4x^3\right)$$

attempt to use the power rule ***(M1)***

$$\log_3 \sqrt{x} = \log_3 \sqrt{2} + \log_3\left(4x^3\right)$$

attempt to use product or quotient rule for logs, $\ln a + \ln b = \ln ab$ ***(M1)***

$$\log_3 \sqrt{x} = \log_3\left(4\sqrt{2}x^3\right)$$

| |
|---|
| **Note:** The ***M*** marks are for attempting to use the relevant log rule and may be applied in any order and at any time during the attempt seen**.** |

$$\sqrt{x} = 4\sqrt{2}x^3$$

$$x = 32x^6$$

$$x^5 = \frac{1}{32}$$ ***(A1)***

$$x = \frac{1}{2}$$ ***A1***

***[5 marks]***

**4.** (a) valid approach to find $\mathrm{P}(R)$ ***(M1)***

tree diagram (must include probabilty of picking box) with correct required probabilities

OR $\mathrm{P}(R \cap B_1) + \mathrm{P}(R \cap B_2)$ OR $\mathrm{P}(R \mid B_1)\mathrm{P}(B_1) + \mathrm{P}(R \mid B_2)\mathrm{P}(B_2)$

$\frac{5}{7} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{1}{2}$ ***(A1)***

$\mathrm{P}(R) = \frac{9}{14}$ ***A1***

***[3 marks]***

(b) events $A$ and $R$ are not independent, since $\frac{9}{14} \cdot \frac{1}{2} \neq \frac{5}{14}$ OR $\frac{5}{7} \neq \frac{9}{14}$ OR $\frac{5}{9} \neq \frac{1}{2}$

**OR** an explanation e.g. different number of red balls in each box ***A2***

**Note**: Both conclusion and reasoning are required. Do not split the ***A2***.

***[2 marks]***

***Total [5 marks]***

**5.** (a) $f'(4) = 6$ ***A1***

***[1 mark]***

(b) $f(4) = 6 \times 4 - 1 = 23$ ***A1***

***[1 mark]***

(c) $h(4) = f\left(g(4)\right)$ ***(M1)***

$h(4) = f(4^2 - 3 \times 4) = f(4)$

$h(4) = 23$ ***A1***

***[2 marks]***

(d) attempt to use chain rule to find $h'$ ***(M1)***

$f'\left(g\left(x\right)\right) \times g'\left(x\right)$ **OR** $\left(x^2 - 3x\right)' \times f'\left(x^2 - 3x\right)$

$h'(4) = \left(2 \times 4 - 3\right) f'\left(4^2 - 3 \times 4\right)$ ***A1***

$= 30$

$y - 23 = 30\left(x - 4\right)$ OR $y = 30x - 97$ ***A1***

***[3 marks]***

***Total [7 marks]***

**6.** (a) **METHOD 1**

attempt to write all LHS terms over a common denominator of $x-1$ ***(M1)***

$$2x-3-\frac{6}{x-1}=\frac{2x(x-1)-3(x-1)-6}{x-1} \text{ OR } \frac{(2x-3)(x-1)}{x-1}-\frac{6}{x-1}$$

$$=\frac{2x^2-2x-3x+3-6}{x-1} \text{ OR } \frac{2x^2-5x+3}{x-1}-\frac{6}{x-1}$$ ***A1***

$$=\frac{2x^2-5x-3}{x-1}$$ ***AG***

***[2 marks]***

**METHOD 2**

attempt to use algebraic division on RHS ***(M1)***

correctly obtains quotient of $2x-3$ and remainder $-6$ ***A1***

$=2x-3-\dfrac{6}{x-1}$ as required. ***AG***

***[2 marks]***

*continue…*

*Question 6 continued.*

(b) consider the equation $\frac{2\sin^2 2\theta - 5\sin 2\theta - 3}{\sin 2\theta - 1} = 0$ ***(M1)***

$\Rightarrow 2\sin^2 2\theta - 5\sin 2\theta - 3 = 0$

**EITHER**

attempt to factorise in the form $(2\sin 2\theta + a)(\sin 2\theta + b)$ ***(M1)***

| **Note:** Accept any variable in place of $\sin 2\theta$. |
|---|

$(2\sin 2\theta + 1)(\sin 2\theta - 3) = 0$

**OR**

attempt to substitute into quadratic formula ***(M1)***

$\sin 2\theta = \frac{5 \pm \sqrt{49}}{4}$

**THEN**

$\sin 2\theta = -\frac{1}{2}$ or $\sin 2\theta = 3$ ***(A1)***

| ***Note:*** Award ***A1*** for $\sin 2\theta = -\frac{1}{2}$ only. |
|---|

one of $\frac{7\pi}{6}$ OR $\frac{11\pi}{6}$ (accept $210$ or $330$) ***(A1)***

$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$ (must be in radians) ***A1***

| ***Note:*** Award ***A0*** if additional answers given. |
|---|

***[5 marks]***

***Total [7 marks]***

**7.** (a) attempt to use discriminant $b^2-4ac(>0)$ ***M1***

$(2p)^2-4(3p)(1-p)(>0)$

$16p^2-12p(>0)$ ***(A1)***

$p(4p-3)(>0)$

attempt to find critical values $\left(p=0,\ p=\frac{3}{4}\right)$ ***M1***

recognition that discriminant $>0$ ***(M1)***

$p<0$ or $p>\frac{3}{4}$ ***A1***

| **Note:** Condone 'or' replaced with 'and', a comma, or no separator |
|---|

***[5 marks]***

(b) $p=4\Rightarrow 12x^2+8x-3=0$

valid attempt to use $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (or equivalent) ***M1***

$x=\frac{-8\pm\sqrt{208}}{24}$

$x=\frac{-2\pm\sqrt{13}}{6}$

$a=-2$ ***A1***

***[2 marks]***

***Total [7 marks]***

**8.** $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{2y}{x}=\frac{\ln 2x}{x^2}$ ***(M1)***

attempt to find integrating factor ***(M1)***

$\left(\mathrm{e}^{\int \frac{2}{x}\mathrm{d}x}=\mathrm{e}^{2\ln x}\right)=x^2$ ***(A1)***

$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=\ln 2x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right)=\ln 2x$

$x^2y=\int \ln 2x\ \mathrm{d}x$

attempt to use integration by parts ***(M1)***

$x^2y=x\ln 2x-x(+c)$ ***A1***

$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{c}{x^2}$

substituting $x=\frac{1}{2}, y=4$ into an integrated equation involving $c$ ***M1***

$4=0-2+4c$

$\Rightarrow c=\frac{3}{2}$

$y=\frac{\ln 2x}{x}-\frac{1}{x}+\frac{3}{2x^2}$ ***A1***

***[7 marks]***

**9.** (a) attempt to expand binomial with negative fractional power ***(M1)***

$$\frac{1}{\sqrt{1+ax}}=(1+ax)^{-\frac{1}{2}}=1-\frac{ax}{2}+\frac{3a^2x^2}{8}+\dots$$ ***A1***

$$\sqrt{1-x}=(1-x)^{\frac{1}{2}}=1-\frac{x}{2}-\frac{x^2}{8}+\dots$$ ***A1***

$$\frac{1}{\sqrt{1+ax}}-\sqrt{1-x}=\frac{(1-a)}{2}x+\left(\frac{3a^2+1}{8}\right)x^2+\dots$$

attempt to equate coefficients of $x$ or $x^2$ ***(M1)***

$$x\ :\quad \frac{1-a}{2}=4b;\ x^2\ :\ \frac{3a^2+1}{8}=b$$

attempt to solve simultaneously ***(M1)***

$$a=-\frac{1}{3},b=\frac{1}{6}$$ ***A1***

***[6 marks]***

(b) $|x|<1$ ***A1***

***[1 mark]***

***Total [7 marks]***

## Section B

**10.** (a) (i) valid approach to find turning point ( $v'=0$, $-\frac{b}{2a}$, average of roots) ***(M1)***

$4-6t=0$ OR $-\frac{4}{2(-3)}$ OR $\frac{-\frac{2}{3}+2}{2}$

$t=\frac{2}{3}$ (s) ***A1***

(ii) attempt to integrate $v$ ***(M1)***

$\int v\ \mathrm{d}t=\int\left(4+4t-3t^2\right)\ \mathrm{d}t=4t+2t^2-t^3(+c)$ ***A1A1***

| **Note:** Award ***A1*** for $4t+2t^2$, ***A1*** for $-t^3$. |
|---|

attempt to substitute their $t$ into their solution for the integral ***(M1)***

distance $=4\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^3$

$=\frac{8}{3}+\frac{8}{9}-\frac{8}{27}$ (or equivalent) ***A1***

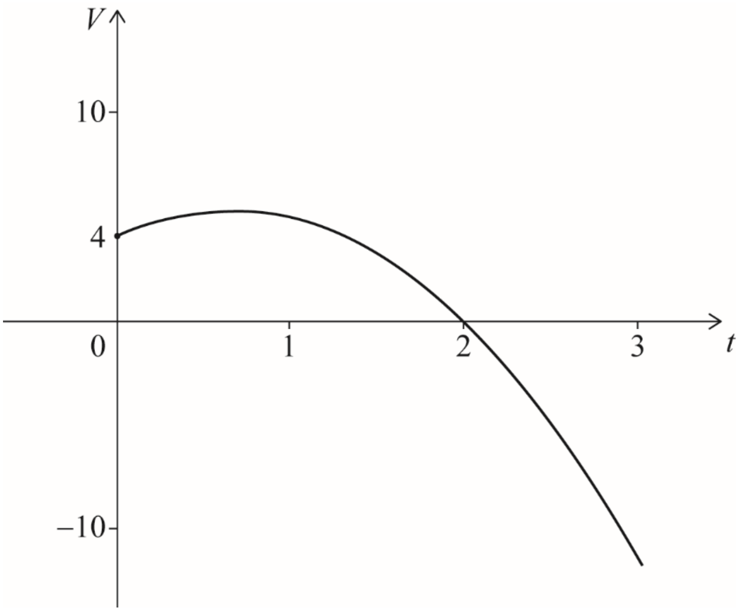
$=\frac{88}{27}$ (m) ***AG***

***[7 marks]***

*continue…*

*Question 10 continued.*

(b)




valid approach to solve $4+4t-3t^2=0$ (may be seen in part (a)) ***(M1)***

$(2-t)(2+3t)$ OR $\frac{-4\pm\sqrt{16+48}}{-6}$

correct $x$- intercept on the graph at $t=2$ ***A1***

**Note**: The following two ***A*** marks may only be awarded if the shape is a concave down parabola. These two marks are independent of each other and the ***(M1)***.

correct domain from 0 to 3 starting at $(0,4)$ ***A1***

**Note:** The 3 must be clearly indicated.

vertex in approximately correct place for $t=\frac{2}{3}$ and $v>4$ ***A1***

***[4 marks]***

*continue…*

*Question 10 continued.*

(c) recognising to integrate between $0$ and $2$, or $2$ and $3$ OR $\int_0^3 \left|4+4t-3t^2\right| \mathrm{d}t$ ***(M1)***

$\int_0^2 \left(4+4t-3t^2\right) \mathrm{d}t$

$=8$ ***A1***

$\int_2^3 \left(4+4t-3t^2\right) \mathrm{d}t$

$=-5$ ***A1***

valid approach to sum the two areas (seen anywhere) ***(M1)***

$\int_0^2 v \ \mathrm{d}t - \int_2^3 v \ \mathrm{d}t$ OR $\int_0^2 v \ \mathrm{d}t + \left|\int_2^3 v \ \mathrm{d}t\right|$

total distance travelled $=13$ (m) ***A1***

***[5 marks]***

***Total [16 marks]***

**11.** (a) For $n=1$

LHS: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right)=x^2\mathrm{e}^x+2x\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

RHS: $\left(x^2+2(1)x+1(1-1)\right)\mathrm{e}^x\left(=\mathrm{e}^x\left(x^2+2x\right)\right)$ ***A1***

so true for $n=1$

now assume true for $n=k$ ; i.e. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right)=\left[x^2+2kx+k(k-1)\right]\mathrm{e}^x$ ***M1***

> **Note**: Do not award ***M1*** for statements such as "let $n=k$". Subsequent marks can still be awarded.

attempt to differentiate the RHS ***M1***

$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\left[x^2+2kx+k(k-1)\right]\mathrm{e}^x\right)$

$=(2x+2k)\mathrm{e}^x+\left(x^2+2kx+k(k-1)\right)\mathrm{e}^x$ ***A1***

$=\left[x^2+2(k+1)x+k(k+1)\right]\mathrm{e}^x$ ***A1***

so true for $n=k$ implies true for $n=k+1$

therefore $n=1$ true and $n=k$ true $\Rightarrow$ $n=k+1$ true

therefore, true for all $n\in\mathbb{Z}^+$ ***R1***

> **Note:** Award **R1** only if three of the previous four marks have been awarded

***[7 marks]***

*continue…*

*Question 11 continued.*

(b) **METHOD 1**

attempt to use $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)=\left[x^2+2nx+n(n-1)\right]\mathrm{e}^x$ ***(M1)***

**Note*:*** For $x=0$, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2\mathrm{e}^x\right)_{|x=0}=n(n-1)$ may be seen.

$f(0)=0, f'(0)=0, f''(0)=2, f'''(0)=6, f^{(4)}(0)=12$

use of $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+\ldots$ ***(M1)***

$\Rightarrow f(x)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

***[3 marks]***

**METHOD 2**

'$x^2\times$ Maclaurin series of $\mathrm{e}^x$' ***(M1)***

$x^2\left(1+x+\frac{x^2}{2!}+\ldots\right)$ ***(A1)***

$\Rightarrow f(x)\approx x^2+x^3+\frac{1}{2}x^4$ ***A1***

***[3 marks]***

*continue…*

*Question 11 continued.*

(c) **METHOD 1**

attempt to substitute $x^2e^x \approx x^2 + x^3 + \frac{1}{2}x^4$ into $\frac{\left(x^2e^x - x^2\right)^3}{x^9}$ ***M1***

$$\frac{\left(x^2e^x - x^2\right)^3}{x^9} \approx \frac{\left(x^2 + x^3 + \frac{1}{2}x^4(+\ldots) - x^2\right)^3}{x^9}$$ ***(A1)***

**EITHER**

$$= \frac{\left(x^3 + \frac{1}{2}x^4 + \ldots\right)^3}{x^9}$$ ***A1***

$$= \frac{x^9(+\text{higher order terms})}{x^9}$$

**OR**

$$\left(\frac{x^3 + \frac{1}{2}x^4(+\ldots)}{x^3}\right)^3$$ ***A1***

$$\left(1 + \frac{1}{2}x(+\ldots)\right)^3$$

**THEN**

$= 1$ $(+$ higher order terms)

So $\lim_{x \to 0}\left[\frac{\left(x^2e^x - x^2\right)^3}{x^9}\right] = 1$ ***A1***

***[4 marks]***

*continue…*

*Question 11 continued.*

**METHOD 2**

$$\lim_{x\to 0}\left[\frac{\left(x^2e^x - x^2\right)^3}{x^9}\right] = \lim_{x\to 0}\left(\frac{x^2e^x - x^2}{x^3}\right)^3$$ ***M1***

$$= \lim_{x\to 0}\left(\frac{e^x - 1}{x}\right)^3$$ ***(A1)***

attempt to use L'Hôpital's rule ***M1***

$$= \lim_{x\to 0}\left(\frac{e^x - 0}{1}\right)^3$$

$$= \left[\lim_{x\to 0} e^x\right]^3$$

$$= 1$$ ***A1***

***[4 marks]***

***Total [14 marks]***

**12.** (a) (i) $\left(1+e^{i\frac{\pi}{6}}-1\right)^3$

$=\left(e^{i\frac{\pi}{6}}\right)^3$ ***A1***

$=e^{i\frac{\pi}{2}}$ ***A1***

$=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

$=i$ ***AG***

| **Note:** Candidates who solve the equation correctly can be awarded the above two marks. The working for part (i) may be seen in part (ii). |
|---|

(ii) $(z-1)^3=e^{i\left(\frac{\pi}{2}+2\pi k\right)}$ ***(M1)***

$z-1=e^{i\left(\frac{\pi}{6}+\frac{4\pi k}{6}\right)}$ ***(M1)***

$(k=1)\Rightarrow\omega_2=1+e^{i\frac{5\pi}{6}}$ ***A1***

$(k=2)\Rightarrow\omega_3=1+e^{i\frac{9\pi}{6}}$ ***A1***

***[6 marks]***

*continue…*

*Question 12 continued.*

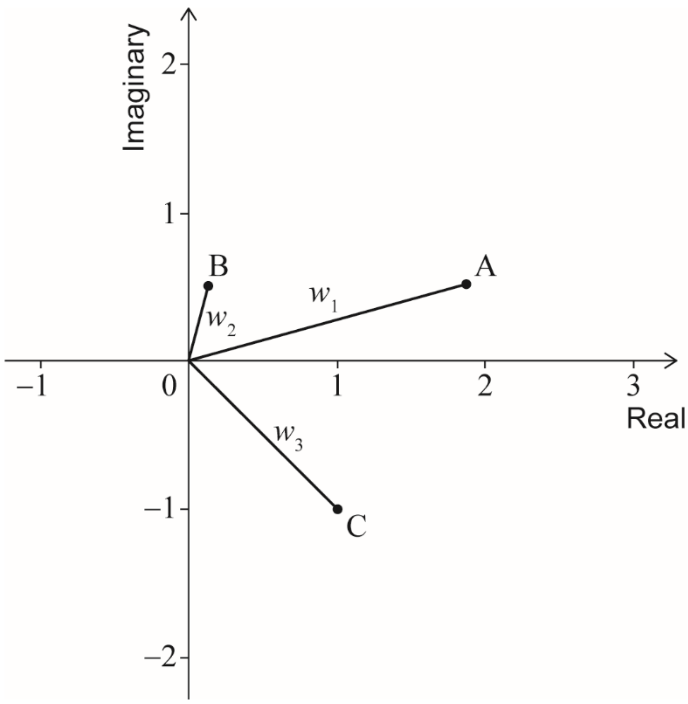
(b) **EITHER**

attempt to express $e^{i\frac{\pi}{6}}$, $e^{i\frac{5\pi}{6}}$, $e^{i\frac{9\pi}{6}}$ in Cartesian form and translate 1 unit in the positive direction of the real axis ***(M1)***

**OR**

attempt to express $w_1, w_2$ and $w_3$ in Cartesian form ***(M1)***

**THEN**




| |
|---|
| **Note:** To award ***A*** marks, it is not necessary to see A,B or C, the $w_i$ , or the solid lines |

***A1A1A1***

***[4 marks]***

*continue…*

*Question 12 continued.*

(c) valid attempt to find $\omega_1 - \omega_3$ (or $\omega_3 - \omega_1$) ***M1***

$$\omega_1 - \omega_3 = \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) - (1 - \mathrm{i}) = \frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i} \text{ OR } \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{2}$$

valid attempt to find $\left|\frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}\right|$ ***M1***

$$= \sqrt{\frac{3}{4} + \frac{9}{4}}$$

$\mathrm{AC} = \sqrt{3}$ ***A1***

***[3 marks]***

*continue…*

*Question 12 continued.*

(d) **METHOD 1**

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(\frac{z-1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$\frac{\alpha-1}{\alpha} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

| **Note:** This step to change from $z$ to $\alpha$ may occur at any point in MS. |
|---|

$\alpha - 1 = \alpha \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$

$\alpha - \alpha \mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = 1$

$\alpha\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) = 1$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

*continue…*

*Question 12 continued.*

**METHOD 2**

$(z-1)^3 = \mathrm{i}z^3 \Rightarrow \left(\frac{z-1}{z}\right)^3 = \mathrm{i}$ ***M1***

$\left(1-\frac{1}{z}\right)^3 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ ***A1***

$1-\frac{1}{z} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ ***A1***

| **Note:** This step to change from $z$ to $\alpha$ may occur at any point in MS. |
|---|

$1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} = \frac{1}{\alpha}$

$\alpha = \frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ ***AG***

*continue…*

*Question 12 continued.*

**METHOD 3**

LHS= $(z-1)^3 = \left(\frac{1}{1-e^{i\frac{\pi}{6}}} - 1\right)^3$

$= \left(\frac{e^{i\frac{\pi}{6}}}{1-e^{i\frac{\pi}{6}}}\right)^3$

$= \frac{i}{\left(1-e^{i\frac{\pi}{6}}\right)^3} \quad \left(= \frac{i}{\frac{5}{2} - \frac{3\sqrt{3}}{2} + i\left(\frac{3\sqrt{3}}{2} - \frac{5}{2}\right)}\right)$ ***M1A1***

| |
|---|
| **Note:** Award ***M1*** for applying de Moivre's theorem (may be seen in modulus- argument form.) |

RHS= $iz^3 = i\left(\frac{1}{1-e^{i\frac{\pi}{6}}}\right)^3$

$= \frac{i}{\left(1-e^{i\frac{\pi}{6}}\right)^3}$ ***A1***

$(z-1)^3 = iz^3$ ***AG***

*continue…*

*Question 12 continued.*

**METHOD 4**

$(z-1)^3 = \mathrm{i}z^3$

$z^3 - 3z^2 + 3z - 1 = \mathrm{i}z^3$

$(1-\mathrm{i})z^3 - 3z^2 + 3z - 1 = 0$ ***(M1)***

$$(1-\mathrm{i})\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^3 - 3\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right)^2 + 3\left(\frac{1}{1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}\right) - 1$$

$$= (1-\mathrm{i}) - 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) + 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^2 - \left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)^3$$ ***(A1)***

$$= (1-\mathrm{i}) - 3\left(1-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right) + 3\left(1-2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right) - \left(1-3\mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + 3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} - \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right)$$ ***A1***

$= 0$ ***AG***

**Note:** If the candidate does not interpret their conclusion, award ***(M1)(A1)A0*** as appropriate.

***[3 marks]***

*continue…*

*Question 12 continued.*

(e) **METHOD 1**

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

$$=\frac{2}{2-\sqrt{3}-i}$$ ***A1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{4-2\sqrt{3}+2i}{\left(2-\sqrt{3}\right)^2+1}$$ ***A1***

$$=\frac{4-2\sqrt{3}+2i}{8-4\sqrt{3}}$$ ***A1***

$$=\frac{1}{2}+\frac{1}{4-2\sqrt{3}}i$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

**Note:** Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

***[6 marks]***

*continue…*

*Question 12 continued.*

**METHOD 2**

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1}{1-\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}$$ ***M1***

attempt to use conjugate to rationalise ***M1***

$$=\frac{1}{\left(1-\cos\frac{\pi}{6}\right)-i\sin\frac{\pi}{6}}\times\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{\left(1-\cos\frac{\pi}{6}\right)^2+\sin^2\frac{\pi}{6}}$$ ***A1***

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{1-2\cos\frac{\pi}{6}+\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}}$$

$$=\frac{\left(1-\cos\frac{\pi}{6}\right)+i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

$$=\frac{1}{2}+\frac{i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$

$$\Rightarrow \mathrm{Re}(\alpha)=\frac{1}{2}$$ ***A1***

**Note**: Their final imaginary part does not have to be correct in order for the final three ***A*** marks to be awarded

***[6 marks]***

*continue…*

*Question 12 continued.*

**METHOD 3**

attempt to multiply through by $-\dfrac{e^{-i\frac{\pi}{12}}}{e^{-i\frac{\pi}{12}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=-\frac{e^{-i\frac{\pi}{12}}}{e^{i\frac{\pi}{12}}-e^{-i\frac{\pi}{12}}}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=-\frac{\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)}{\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}-\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)}$$ ***A1***

$$=-\frac{\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}}{2i\sin\frac{\pi}{12}}$$ ***A1***

$$=\frac{1}{2}-\frac{1}{2i}\cot\frac{\pi}{12}\ \left(=\frac{1}{2}+\frac{1}{2}i\cot\frac{\pi}{12}\right)$$

$$\Rightarrow \operatorname{Re}(\alpha)=\frac{1}{2}$$ ***A1***

***[6 marks]***

*continue…*

*Question 12 continued.*

**METHOD 4**

attempt to multiply through by $\frac{1-e^{-i\frac{\pi}{6}}}{1-e^{-i\frac{\pi}{6}}}$ ***M1***

$$\frac{1}{1-e^{i\frac{\pi}{6}}}=\frac{1-e^{-i\frac{\pi}{6}}}{1-e^{-i\frac{\pi}{6}}-e^{i\frac{\pi}{6}}+1}$$ ***A1***

attempting to re-write in r-cis form ***M1***

$$=\frac{1-\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2-2\cos\frac{\pi}{6}}$$ ***A1***

attempt to re-write in Cartesian form ***M1***

$$=\frac{1-\frac{\sqrt{3}}{2}-\frac{1}{2}i}{2-\sqrt{3}}\left(=\frac{\frac{2-\sqrt{3}}{2}}{2-\sqrt{3}}+i\frac{\frac{1}{2}}{2-\sqrt{3}}\right)$$

$$\Rightarrow \text{Re}(\alpha)=\frac{1}{2}$$ ***A1***

| |
|---|
| **Note:** Their final imaginary part does not have to be correct in order for the final ***A*** mark to be awarded |

***[6 marks]***

***Total [22 marks]***